MARTIN'S AXIOM AND THE TRANSITIVITY OF $P_{\mathfrak{c}}$-POINTS

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ABSTRACT

It is shown to be consistent with Martin's Axiom and $2^{\aleph_0} > \aleph_1$ that every two $P_{\mathfrak{c}}$-points in $\beta\mathbb{N} \setminus \mathbb{N}$ have the same topological type.

1. Introduction

Two points, p and q, in $\beta\mathbb{N} \setminus \mathbb{N}$ will be said to be **topologically equivalent** or, to have the same **topological type** if and only if there is an autohomeomorphism

$$\Phi : \beta\mathbb{N} \setminus \mathbb{N} \mapsto \beta\mathbb{N} \setminus \mathbb{N}$$

such that $\Phi(p) = q$. It was shown by W. Rudin in [2] that, assuming CH, every two P-points are topologically equivalent. In [1] van Mill asked whether the same result can be obtained for $P_{\mathfrak{c}}$-points by assuming MA instead of CH. That the answer to this question is negative follows from the fact that it is consistent with MA that every autohomeomorphism of $\beta\mathbb{N} \setminus \mathbb{N}$ is induced by an almost permutation of ω [4]. Since MA implies that there are the maximum number of $P_{\mathfrak{c}}$-points while there are are only $2^{\aleph_0}$ almost permutations, topological equivalence classes all have size $2^{\aleph_0}$ and so there are many $P_{\mathfrak{c}}$-points which are not topologically equivalent.

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This raises the question of whether or not MA itself implies that there are two $P_{\mathfrak{c}}$-points in $\beta\mathbb{N}\setminus\mathbb{N}$ which are not topologically equivalent. It is the purpose of this paper to show that the answer is negative thereby providing the complete solution to van Mill's question. In particular, it will be shown that there is an equivalence relation on $\beta\mathbb{N}\setminus\mathbb{N}$ which, under MA, has only one equivalence class of $P_{\mathfrak{c}}$-points, and that it is consistent with MA that any two $P_{\mathfrak{c}}$-points which are equivalent with respect to this relation are topologically equivalent. The restriction to $P_{\mathfrak{c}}$-points is of course important since MA and $2^{\aleph_0} > \aleph_1$ implies that there are both P_{ω_2}-points and P_{ω_1}-points which are not P_{ω_2}-points and these are distinct topological properties.

The notation and terminology of this paper will adhere as much as possible to accepted standards but some of the main points are listed here. The relation $a \subseteq^* b$ means that $|a\setminus b| < \aleph_0$. A P_κ-filter is a filter on ω, p, such that for every $\mathcal{A} \in [p]^{<\kappa}$ there is $B \in p$ such that $B \subseteq^* A$ for every $A \in \mathcal{A}$ and, moreover, κ is the greatest cardinal with this property. A P_κ-point is an ultrafilter which is a P_κ-filter. If p is a filter then p^* will denote the dual ideal to p.

2. An equivalence relation on $P_{\mathfrak{c}}$-points

In this section a relation on ultrafilters will be defined and particular attention will be paid to this relation on $P_{\mathfrak{c}}$-points. This relation will be used in obtaining the main result of this paper; whether or not it is interesting in its own right is left to the reader to decide. To begin, a preliminary piece of notation is required.

Definition 2.1: For any partial one-to-one function $h : \omega \to \omega$ define

$$\Sigma(h) = \{n \in \omega; (\forall i \in n)(\{h(i), h^{-1}(i)\} \subseteq n)\}.$$

For any infinite $A \subseteq \omega$ and $B \subseteq \omega$ define $\Phi_{A,B}$ to be the unique order preserving bijection from A to B and define $\Sigma(A,B) = \Sigma(\Phi_{A,B})$. ∎

The set $\Sigma(h)$ can be thought of as the set of points where h reflects. Notice that not all functions reflect — for example, the function $h(n) = n + 1$ does not — and so it is possible that $\Sigma(h) = \emptyset$. It is also easy to see that $\Sigma(A,B) = \{n \in \omega; |A \cap n| = |B \cap n|\}$. In proving Lemma 4.1 the following variant of the Galvin–McKenzie game will be used.

Definition 2.2: The game $G(p,q,r)$ is defined for each triple of filters p, q and r on ω. Play alternates between Players I and II. Player I chooses $A_n \in p$,

$B_n \in q$ and $C_n \in r$ while Player II chooses $a_n \in [A_n]^{<\aleph_0}$, $b_n \in [B_n]^{<\aleph_0}$ and $c_n \in C_n$ subject to the condition that $\max(a_n \cup b_n) < c_n < \min(a_{n+1} \cup b_{n+1})$. Player II is declared to be the winner if $\bigcup\{a_n; n \in \omega\} \in p$, $\bigcup\{b_n; n \in \omega\} \in q$ and $\{c_n; n \in \omega\} \in r$. The game $G(p,q)$ is defined similarly for each pair of filters p and q by omitting any mention of r. To be precise, Player I chooses $A_n \in p$ and $B_n \in q$ while Player II chooses $a_n \in [A_n]^{<\aleph_0}$ and $b_n \in [B_n]^{<\aleph_0}$. Player II is declared to be the winner if $\bigcup\{a_n; n \in \omega\} \in p$ and $\bigcup\{b_n; n \in \omega\} \in q$. ∎

The following lemma is a straightforward generalisation of an unpublished result of Galvin and McKenzie, but a proof is included for the reader's convenience.

LEMMA 2.1: *If p and q are P-points then Player* I *has no winning strategy in the game $G(p,q)$.*

Proof: Suppose that Player I does have a winning strategy. Since the strategy is necessarily a countable object it is possible to choose $A \in p$ and $B \in q$ such that if the pair of sets X and Y is ever chosen by Player I in any game by this strategy then $A \subseteq^* X$ and $B \subseteq^* Y$.

Now construct a sequence of integers $\{N_i; i \in \omega\}$ such that $A \setminus X \subseteq N_{i+1}$ and $B \setminus Y \subseteq N_{i+1}$ for any of the finitely many plays of the game $G(p,q)$ in which Player II has chosen only non-empty sets contained in N_i and Player I's strategy advises to choose the pair $X \in p$ and $Y \in q$. Choose $i \in 3$ such that

$$\bigcup\{N_{3n+i+1} \setminus N_{3n+i}; n \in \omega\} \in p$$

and then choose $i' \in 3$ such that $\bigcup\{N_{3n+i'+1} \setminus N_{3n+i'}; n \in \omega\} \in q$. Let $j \in 3 \setminus \{i, i'\}$. It now follows that if the game $G(p,q)$ is played, with Player I following the supposed winning strategy and Player II playing $(N_{3k+j+3} \setminus N_{3k+j+1}) \cap A$ and $(N_{3k+j+3} \setminus N_{3k+j+1}) \cap B$ at stage k of the game, then each of these moves is legal and Player II wins. ∎

Definition 2.3: Define $p \bowtie_r q$ if and only if p and q are ultrafilters and r is a filter on ω and

1. for every $X \in p$ there is $Y \in q$ such that $\Sigma(X,Y) \in r$,
2. for every $X \in q$ there is $Y \in p$ such that $\Sigma(X,Y) \in r$,
3. Player I does not have a winning strategy in the game $G(p,q,r)$.

The relation $\bowtie$ will be defined to be the transitive closure of $\bigcup\{\bowtie_r; r \in \beta\mathbb{N} \setminus \mathbb{N}\}$. ∎

It is easy to see that $\bowtie$ is symmetric and it is transitive by definition. It is also reflexive because $p \bowtie_r p$ holds for any two ultrafilters p and r since $\Sigma(A, A) = \omega$ for any $A \subseteq \omega$. Hence $\bowtie$ is an equivalence relation. The first point worth noting is that Martin's Axiom implies that all $P_{\mathfrak{c}}$-points are $\bowtie$-equivalent. This will be used to show that all $P_{\mathfrak{c}}$-points are topologically equivalent.

LEMMA 2.2: *MA implies that if p and q are $P_{\mathfrak{c}}$-points then there are $P_{\mathfrak{c}}$-filters r, r' and a $P_{\mathfrak{c}}$-point t such that $p \bowtie_r t$ and $t \bowtie_{r'} q$. In particular, $p \bowtie q$.*

Proof: Let $\{A_\xi; \xi \in \mathfrak{c}\}$ be a $\subseteq^*$-descending base for p and $\{B_\xi; \xi \in \mathfrak{c}\}$ be a $\subseteq^*$-descending base for q. Also, let $\{W_\xi; \xi \in \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega$ and let $\{S_\xi; \xi \in \mathfrak{c}\}$ enumerate, cofinally often, all possible strategies for Player I in any game $G(p', q', r')$ — that this is possible follows from the fact that each strategy is a countably branching tree, because the branching is determined by Player II's moves which are simply finite sets of integers. It suffices to construct descending towers $\{C_\xi; \xi \in \mathfrak{c}\}$, $\{D_\xi; \xi \in \mathfrak{c}\}$ and $\{E_\xi; \xi \in \mathfrak{c}\}$ such that, letting c_α, d_α and e_α refer to the filters generated by the first α sets from $\{C_\xi; \xi \in \mathfrak{c}\}$, $\{D_\xi; \xi \in \mathfrak{c}\}$ and $\{E_\xi; \xi \in \mathfrak{c}\}$ respectively, the following conditions hold:

- there is $\overline{A}_\xi \in p$ such that $\overline{A}_\xi \subseteq A_\xi$ and $\Sigma(\overline{A}_\xi, E_\xi) \supseteq C_\xi$,
- there is $\overline{B}_\xi \in q$ such that $\overline{B}_\xi \subseteq B_\xi$ and $\Sigma(\overline{B}_\xi, E_\xi) \supseteq D_\xi$,
- $E_\xi \subseteq^* W_\xi$ or $E_\xi \cap W_\xi =^* \emptyset$,
- if S_ξ is a strategy for Player I in the game $G(p, e_\xi, c_\xi)$ then it is not a winning strategy in the game $G(p, e_{\xi+1}, c_{\xi+1})$,
- if S_ξ is a strategy for Player I in the game $G(q, e_\xi, d_\xi)$ then it is not a winning strategy in the game $G(q, e_{\xi+1}, d_{\xi+1})$,

because the filters generated by $\{C_\xi; \xi \in \mathfrak{c}\}$ and $\{D_\xi; \xi \in \mathfrak{c}\}$ are $P_{\mathfrak{c}}$-filters and $r = c_{\mathfrak{c}}$, $r' = d_{\mathfrak{c}}$ and $t = e_{\mathfrak{c}}$ satisfy the conclusion of the lemma. The fact that $p \bowtie_r t$ follows from the first condition and the fact (easily proved) that if $\Sigma(\overline{A}_\xi, E_\xi) \supseteq C_\xi$ then there is some $E' \supseteq E_\xi$ such that $\Sigma(A_\xi, E') \supseteq C_\xi$. A similar argument applies to show that $t \bowtie_{r'} q$.

To see that the construction can be completed, suppose that descending towers $\{C_\xi; \xi \in \alpha\}$, $\{D_\xi; \xi \in \alpha\}$ and $\{E_\xi; \xi \in \alpha\}$ have been constructed satisfying the required hypothesis. First consider the case that $\alpha = \beta + 1$.

Two tasks must be accomplished: A winning play against the strategy S_β must be found for Player II and it must be decided whether or not the set W_β will belong to the filter $e_{\alpha+1}$. Assume that $W_\beta \cap E_\beta$ is infinite since otherwise $E_\beta \subseteq^* \omega \setminus W_\beta$. From the strategy S_β it is possible to construct an associated strategy S^*_β in the game $G(p,q)$. To do this, let S^p_β denote the strategy S_β when considered as a strategy in the game $G(p, e_\alpha, c_\alpha)$ and let S^q_β denote the strategy S_β when considered as a strategy in the game $G(q, e_\alpha, d_\alpha)$ — if it happens that the strategy S_β does not make sense in either of these games (that is, it mentions sets other than those from p, c_α and e_α or from q, d_α and e_α) then let S^p_β or S^q_β be an arbitrary strategy in the appropriate game. Suppose that a sequence $\{((A^i, B^i), (a_i, b_i)); i \in k\}$ has been played in the game $G(p,q)$ — in other words, Player I has chosen $\{(A^i, B^i); i \in k\}$ while Player II has chosen $\{(a_i, b_i); i \in k\}$ — and that Player I has also chosen auxiliary sets $\{(C^i, D^i, E^i_p, E^i_q, c^i, d^i, e^i_p, e^i_q, a'_i, b'_i); i \in k-1\}$ such that:

- $C^i \in c_\beta$, $D^i \in d_\beta$, $E^i_q \in e_\beta$ and $E^i_p \in e_\beta$,
- $c^i < d^i < c^{i+1}$,
- $C^{i+1} \subseteq C^i$, $D^{i+1} \subseteq D^i$, $E^{i+1}_q \subseteq E^i_q$ and $E^{i+1}_p \subseteq E^i_p$,
- $c^i \in C_\beta$ and $d^i \in D_\beta$,
- $C^{i+1} \cap c^i = \emptyset$ and $D^{i+1} \cap d^i = \emptyset$,
- $|C_\beta \cap C^i| = \aleph_0$, $|D_\beta \cap D^i| = \aleph_0$ and $|W_\beta \cap E_\beta \cap E^i_p \cap E^i_q| = \aleph_0$,
- $E^{i+1}_p \cap \max(e^i_p \cap e^i_q) = \emptyset$ and $E^{i+1}_q \cap \max(e^i_p \cap e^i_q) = \emptyset$,
- $a_i \cup a'_i \subseteq A^i \cap (c^i \setminus c^{i-1})$ and $b_i \cup b'_i \subseteq B^i \cap (d^i \setminus d^{i-1})$,
- $a'_i \subseteq A^i \cap A_\beta$ and $b'_i \subseteq B^i \cap B_\beta$,
- $e^i_q \subseteq (d^i \setminus d^{i-1}) \cap E_\beta \cap W_\beta \cap E^i_q$ and $e^i_p \subseteq (c^i \setminus c^{i-1}) \cap E_\beta \cap W_\beta \cap E^i_p$,
- $e^i_p \cap (c^i \setminus d^{i-1}) = e^i_q \cap (c^i \setminus d^{i-1})$,
- $e^{i+1}_p \cap (d^i \setminus c^i) = e^i_q \cap (d^i \setminus c^i)$,

- $|e^i_p \cap (c^i \setminus c^{i-1})| = |(a_i \cup a'_i) \cap A_\beta \cap (c^i \setminus c^{i-1})|$,
- $|e^i_q \cap (d^i \setminus d^{i-1})| = |(b_i \cup b'_i) \cap B_\beta \cap (d^i \setminus d^{i-1})|$.

Suppose also that Player I has also chosen C^{k-1}, D^{k-1}, E^{k-1}_p, E^{k-1}_q and a'_{k-1} and that Player II has just chosen (a_{k-1}, b_{k-1}). The strategy S^*_β will tell Player I how to choose (A^k, B^k), $(C^k, D^k, E^k_p, E^k_q, a'_k)$ and $(c^{k-1}, d^{k-1}, e^{k-1}_p, e^{k-1}_q, b'_k)$ based on the sets played by Player I and Player II so far as well as the auxiliary sets chosen by Player I. In particular, Player I should choose (A^k, B^k), $(C^k, D^k, E^k_p, E^k_q, a'_k)$ and $(c^{k-1}, d^{k-1}, e^{k-1}_p, e^{k-1}_q, b'_k)$ as follows:

- Player I defines c^{k-1} to be the least element of
$$C_\beta \cap C^{k-1} \setminus (\max(a_{k-1} \cup a'_{k-1}) \cup d^{k-2})$$
such that
$$|c^{k-1} \cap E_\beta \cap E^{k-1}_p \cap E^{k-1}_q \cap W_\beta \setminus d^{k-2}| \geq |a_{k-1} \cup a'_{k-1}|;$$
- Player I then chooses $e'_{k-1} \subseteq E_\beta \cap E^{k-1}_p \cap E^{k-1}_q \cap W_\beta \cup (c^{k-1} \setminus d^{k-2})$ such that
$$|e'_{k-1}| + |e^{k-2}_q \setminus c^{k-2}| = |(a'_{k-1} \cup a_{k-1}) \cap A_\beta|$$
noting that, by the induction hypothesis, $e^{k-2}_q \setminus c^{k-2} = e^{k-2}_q \cap d_{k-2} \setminus c^{k-2}$;
- next, Player I chooses $b'_{k-1} \subseteq B_\beta \setminus d^{k-2}$ such that $|b'_{k-1}| \geq |e'_{k-1}|$;
- then d^{k-1} is defined to be the least element of
$$D_\beta \cap D^{k-1} \setminus (\max(b_{k-1} \cup b'_{k-1}) \cup c^{k-1})$$
such that
$$|d^{k-1} \cap E_\beta \cap E^{k-1}_p \cap E^{k-1}_q \cap W_\beta \setminus c^{k-1}| \geq |b_{k-1} \cup b'_{k-1}|;$$
- Player I chooses $e''_{k-1} \subseteq E_\beta \cap E^{k-1}_p \cap E^{k-1}_q \cap W_\beta \cap (d^{k-1} \setminus c^{k-1})$ such that
$$|e''_{k-1}| + |e'_{k-1}| = |(b'_{k-1} \cup b_{k-1}) \cap B_\beta|;$$
- Player I then defines $e^{k-1}_p = e^{k-2}_q \cap (d^{k-2} \setminus c^{k-2}) \cup e'_{k-1}$;
- and $e^{k-1}_q = e^{k-1}_p \cap (c^{k-1} \setminus d^{k-2}) \cup e''_{k-1} = e'_{k-1} \cup e''_{k-1}$;

- then (A^k, E_p^k, C^k) are chosen by Player I if this is the play dictated by the strategy S_β^p if the sequence

$$\{((A^i, E_p^i, C^i), (a_i, e_p^i, c^i)); i \in k\}$$

had been played in the game $G(p, e_\alpha, c_\alpha)$;

- then (B^k, E_q^k, D^k) are chosen by Player I if this is the play dictated by the strategy S_β^q if the sequence

$$\{((B^i, E_q^i, D^i), (b_i, e_q^i, d^i)); i \in k\}$$

had been played in the game $G(q, e_\alpha, d_\alpha)$ — notice that $|W_\beta \cap E_\beta \cap E_p^k \cap E_q^k| = \aleph_0$ because E_p^k and E_q^k must be chosen from e_α;

- Player I chooses $a'_k \subseteq A^k \cap A_\beta \setminus c^{k-1}$ such that $|a'_k| \geq |e_q^{k-1} \setminus c^{k-1}|$.

It follows from Lemma 2.1 that there is a play of the game $G(p, q)$ where Player I follows the strategy S_β^* but Player II wins. In other words, there is a play $\{((A^k, B^k), (a_k, b_k)); k \in \omega\}$ of the game $G(p, q)$ — as well as auxiliary sets

$$\{(C^i, D^i, E_p^i, E_q^i, c^i, d^i, e_p^i, e_q^i, a'_i, b'_i); i \in k\}$$

— such that $\cup\{a_k; k \in \omega\} \in p$ and $\cup\{b_k; k \in \omega\} \in q$. Furthermore,

$$\{((A^i, E_p^i, C^i), (a_i, e_p^i, c^i)); i \in \omega\}$$

is a legal play of the game $G(p, e_\alpha, c_\alpha)$ in which Player I follows the strategy S_β^p and

$$\{((B^i, E_q^i, D^i), (b_i, e_q^i, d^i)); i \in \omega\}$$

is a legal play of the game $G(q, e_\alpha, d_\alpha)$ in which Player I follows the strategy S_β^q. It can be concluded that, since $\{(A^k, B^k, a_k, b_k); k \in \omega\}$ is a winning play for Player II in the game $G(p, q)$, that if the following definitions are made:

- $\overline{A}_\alpha = \cup\{(a_k \cup a'_k) \cap A_\beta; k \in \omega\}$,
- $\overline{B}_\alpha = \cup\{(b_k \cup b'_k) \cap B_\beta; k \in \omega\}$,
- $C_\alpha = \{c^k; k \in \omega\}$,
- $D_\alpha = \{d^k; k \in \omega\}$,

- $E_\alpha = \cup\{e^k_p; k \in \omega\} =^* \cup\{e^k_q; k \in \omega\}$,

then $\{((A^i, E^i_p, C^i), (a_i, e^i_p, c^i)); i \in \omega\}$ is a winning play of the game $G(p, e_{\alpha+1}, c_{\alpha+1})$ and $\{((B^i, E^i_q, D^i), (b_i, e^i_q, d^i)); i \in \omega\}$ is a winning play of the game $G(q, e_{\alpha+1}, d_{\alpha+1})$ for Player II. It follows that the strategy S_β is not a winning strategy for Player I in $G(p, e, c)$ or $G(p, e, d)$. It has also been decided whether or not W_β will belong to the filter generated by $\{E_\beta; \beta \in \mathfrak{c}\}$. Moreover, Player I's strategy was designed to ensure that $\Sigma(\overline{A}_\alpha, E_\alpha) \supseteq C_\alpha$ and $\Sigma(\overline{B}_\alpha, E_\alpha) \supseteq D_\alpha$ because

$$\begin{aligned} |E^i_p \cap (c^i \setminus c^{i-1})| &= |e^i_p \cap (c^i \setminus c^{i-1})| \\ &= |(a_i \cup a'_i) \cap A_\beta \cap (c^i \setminus c^{i-1})| = |\bar{A}_\alpha \cap (c^i \setminus c^{i-1})| \end{aligned}$$

and

$$\begin{aligned} |E^i_q \cap (d^i \setminus d^{i-1})| &= |e^i_q \cap (d^i \setminus d^{i-1})| \\ &= |(b_i \cup b'_i) \cap B_\beta \cap (d^i \setminus d^{i-1})| = |\bar{B}_\alpha \cap (d^i \setminus d^{i-1})| \ . \end{aligned}$$

Notice that the strategies are enumerated cofinally often so each strategy will eventually be eliminated at some successor stage.

It remains to consider what happens if α is a limit ordinal. In this case there is no need to deal with any strategy or to decide the inclusion of a set into e_α. Simply apply Martin's axiom to the following partial order $\mathbb{P}$ defined by $(e, c, d, \Gamma) \in \mathbb{P}$ if and only if:

- $\Gamma \in [\alpha]^{<\aleph_0}$,
- $\{e, c, d\} \subseteq [\omega]^{<\aleph_0}$,
- $(\forall j \in c)(|A_\alpha \cap j| = |e \cap j|)$,
- $(\forall j \in d)(|B_\alpha \cap j| = |e \cap j|)$.

The ordering on $\mathbb{P}$ is defined by $(e, c, d, \Gamma) \leq (e', c', d', \Gamma')$ if and only if:

- $e \subseteq e'$, $d \subseteq d'$, $c \subseteq c'$ and $\Gamma \subseteq \Gamma'$,
- $e' \setminus e \subseteq \cap\{E_\xi; \xi \in \Gamma\}$.

It is a standard exercise to see that a generic set G on this partial order produces $C'_\alpha = \cup\{c; (e, c, d, \Gamma) \in G\}$, $D'_\alpha = \cup\{d; (e, c, d, \Gamma) \in G\}$ and $E_\alpha = \cup\{e; (e, c, d, \Gamma) \in G\}$, such that

- $E_\alpha \subseteq^* E_\xi$ for each $\xi \in \alpha$,
- $\Sigma(A_\alpha, E_\alpha) \cap C'_\alpha \cap C_\xi$ are $\Sigma(B_\alpha, E_\alpha) \cap D'_\alpha \cap D_\xi$ are infinite for each $\xi \in \alpha$.

The first condition is a direct consequence of the definition of the partial order $\mathbb{P}$ and the fact that any $\xi \in \alpha$ can be added to the Γ of any condition in $\mathbb{P}$. To obtain the second clause suppose that $\xi \in \alpha$, $k \in \omega$ and $(e, c, d, \Gamma) \in \mathbb{P}$. It must be shown that it is possible to extend $(e, c, d, \Gamma) \in \mathbb{P}$ to $(e', c', d, \Gamma) \in \mathbb{P}$ such that $c' \cap C_\xi \not\subseteq k$ (or $(e', c, d', \Gamma) \in \mathbb{P}$ such that $d' \cap D_\xi \not\subseteq k$) — the definition of $\mathbb{P}$ will do the rest. To accomplish this choose $j \in \omega$ such that

- $j > \max\{k, \max(e), \max(c)\}$,
- $|A_\alpha \cap j| \geq |e \setminus \max(c)|$,
- $|(A_\alpha \setminus \max(c)) \cap j| \leq |(\cap_{\gamma\in\Gamma} E_\gamma \setminus \max(c)) \cap j|$,
- $j \in C_\xi$ (or $j \in D_\xi$).

Let $c' = c \cup \{j\}$ (or let $d' = d \cup \{j\}$) and extend e to e' as required. The only question which needs to be answered is why it is possible to choose $j \in C_\xi$ such that $|(A_\alpha \setminus \max(c)) \cap j| \leq |(\cap_{\gamma\in\Gamma} E_\gamma \setminus \max(c)) \cap j|$. The reason is that if $\mu = \max(\Gamma)$ then $\Sigma(A_\mu, \cap_{\gamma\in\Gamma} E_\gamma) \equiv^* C_\mu$ by the induction hypothesis. Moreover, $C_\xi \cap C_\mu$ is infinite and $A_\mu \supseteq^* A_\alpha$.

It is now an easy matter to refine C'_α and D'_α to obtain a tower. ∎

3. A slight modification of Veličković's partial order

It will be shown that it is consistent with MA_{ω_1} that if p and q are $P_\mathfrak{c}$ points and $p \bowtie q$ then p and q have the same topological type. The main step in doing this will depend on the following partial order which is a modification of a partial order used by Veličković [5].

Definition 3.1: For any two ultrafilters p and q and a filter r define the partial order $\mathbb{Q}(p, q, r)$ to consist of all one-to-one functions f such that:

1. $\mathrm{dom}(f) \in p^*$,
2. $\mathrm{ran}(f) \in q^*$,
3. $\Sigma(f) \in r$,

ordered under $\subseteq^*$.

Much of this paper will be devoted to establish closure properties for $\mathbb{Q}(p,q,r)$ under certain conditions. The following lemma will prove to be useful in this context.

LEMMA 3.1: *Suppose that p and q are P_λ-points and r is a P_λ-filter. Suppose also that $\eta \in \lambda$. If $\{f_\xi; \xi \in \eta\}$ is an increasing sequence from $\mathbb{Q}(p,q,r)$ and if there is f' such that $f' \supseteq^* f_\xi$ for each $\xi \in \eta$ then there is $f \in \mathbb{Q}(p,q,r)$ such that $f \supseteq^* f_\xi$ for each $\xi \in \eta$.*

Proof: Choose $C \in r$, $A \in p^*$ and $B \in q^*$ such that $A \supseteq^* \mathrm{dom}(f_\xi)$, $B \supseteq^* \mathrm{ran}(f_\xi)$ and $C \subseteq^* \Sigma(f_\xi)$ for each $\xi \in \eta$. It follows that for each $\xi \in \eta$ there is some $A'_\xi \subseteq \mathrm{dom}(f_\xi) \cap A$ such that

- $|\mathrm{dom}(f_\xi) \setminus A'_\xi| < \aleph_0$,
- $f_\xi \restriction A'_\xi \subseteq f'$,
- $C \subseteq \Sigma(f_\xi \restriction A'_\xi)$.

Let $f'' = \cup\{f_\xi \restriction A'_\xi; \xi \in \eta\}$ noting that $f'' \subseteq f'$. Now let $f = f'' \restriction ({f'}^{-1} B)$. It is easy to see that $\Sigma(f) \supseteq \Sigma(f'') \supseteq \cap_{\xi\in\lambda}\Sigma(f_\xi \restriction A'_\xi) \supseteq C \in r$. ∎

LEMMA 3.2: *If p and q are P_λ-points and r is a P_λ-filter and $\lambda \geq \omega_1$ then $\mathbb{Q}(p,q,r)$ is countably closed.*

Proof: Given a sequence $\{f_n; n \in \omega\} \subseteq \mathbb{Q}(p,q,r)$ such that $f_n \subseteq^* f_{n+1}$ for each $n \in \omega$ choose inductively k_n such that $f_\omega = \cup\{f_n \restriction (\omega \setminus k_n); n \in \omega\}$ is a function and apply Lemma 3.1. ∎

LEMMA 3.3: *If p and q are P_λ-points and r is a P_λ-filter and $p \bowtie_r q$ then $1 \Vdash_{\mathbb{Q}(p,q,r)}$ "q has the same topological type as p".*

Proof: By Lemma 3.2 no reals are added by forcing with $\mathbb{Q}(p,q,r)$ and so p and q are still ultrafilters in the generic extension. It is sufficient to show that there is an automorphism of the Boolean algebra $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ which takes the filter p to the filter q. Let $[X]$ represent the equivalence class of X in $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$. If G is $\mathbb{Q}(p,q,r)$-generic define an automorphism $\Phi_G : \mathcal{P}(\omega)/[\omega]^{<\aleph_0} \mapsto \mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ by

$$\Phi_G([X]) = \begin{cases} [\{g(i); i \in X\}] & \text{if } (\exists g \in G)(X \subseteq \mathrm{dom}(g)), \\ [\omega \setminus \{g(i); i \in \omega \setminus X\}] & \text{if } (\exists g \in G)(\omega \setminus X \subseteq \mathrm{dom}(g)). \end{cases}$$

If it can be shown that $\mathrm{dom}(\Phi_G) = \mathcal{P}(\omega)/[\omega]^{<\aleph_0} = \mathrm{ran}(\Phi_G)$ then it is routine to check that Φ_G induces the desired autohomeomorphism of $\beta\mathbb{N} \setminus \mathbb{N}$. It suffices to check that if $X \subseteq \omega$ is in the ground model and $f \in \mathbb{Q}(p,q,r)$ then there is $f' \supseteq f$ such that $f' \Vdash_{\mathbb{Q}(p,q,r)}$ "$[X] \in \mathrm{dom}(\Phi_G) \cap \mathrm{ran}(\Phi_G)$". To see that this is so assume that $X \in p^*$ (otherwise deal with $\omega \setminus X$). Let $X' = X \cup \mathrm{dom}(f) \in p^*$. The definition of $\bowtie_r$ ensures that there is $Y \in q^*$ such that $\Sigma(X', Y) \in r$. It is an easy matter to extend f to f' so that $X \subseteq \mathrm{dom}(f')$ and $\mathrm{ran}(f') \subseteq \mathrm{ran}(f) \cup Y$. Obviously $f' \Vdash$ "$[X] \in \mathrm{dom}(\Phi_G)$". It is equally easy to put $[X]$ into the range of Φ_G. ∎

4. The iterated partial order

In this section it will be shown how to construct a model of set theory in which any two $P_{\mathfrak{c}}$-points are topologically equivalent. To begin, let V be a model of PFA and, in this model, let $\{(p_\alpha, q_\alpha, r_\alpha); \alpha \in \mathfrak{c}\}$ be an enumeration of all names of triples of $P_{\mathfrak{c}}$-points which come from $\aleph_2$-chain condition partial orders of size $\aleph_2$. In this model construct an iteration $\{\mathbb{P}_\alpha; \alpha \in \omega_3\}$ with supports of size $\aleph_1$ such that, for each $\alpha \in \omega_3$, if $1 \Vdash_{\mathbb{P}_\alpha}$ "$p_\alpha \bowtie_{r_\alpha} q_\alpha$" then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}(p_\alpha, q_\alpha, r_\alpha)$ — otherwise $\mathbb{Q}(p_\alpha, q_\alpha, r_\alpha)$ is trivial. It follows from Lemma 3.2 that $\mathbb{P}_{\omega_3}$ is countably closed. Moreover, since all triples have been enumerated, it follows that

$$1 \Vdash_{\mathbb{P}_{\omega_3}} \text{“}(\forall p, q, r)(\text{ if } p \bowtie_r q \text{ then } (\exists \alpha)(p = p_\alpha, q = q_\alpha \text{ and } r = r_\alpha))\text{”}.$$

If it can be shown that $\mathbb{P}_{\omega_3}$ is $\aleph_2$-distributive then it will immediately follow that MA_{ω_1} also holds in this model since MA_{ω_1} refers only to structures (that is partial orders and families of dense sets) of size $\aleph_1$ and no new such structures have been added. It will then follow from Lemmas 3.3 and 2.2 that the resulting model is one in which all $P_{\mathfrak{c}}$-points are topologically equivalent.

Definition 4.1: Let $\{g_\alpha; \alpha \in \xi\}$ be an increasing sequence in $\mathbb{Q}(p,q,r)$. Define the partial order

$$\mathbb{R}(\{g_\alpha; \alpha \in \xi\}) = \{g; (\exists \alpha \in \xi)(g \equiv^* g_\alpha)\}.$$

The ordering on $\mathbb{R}(\{g_\alpha; \alpha \in \xi\})$ is $\subseteq$ as opposed to $\subseteq^*$ in $\mathbb{Q}(p,q,r)$. ∎

Notice that the definition of $\mathbb{R}(\{g_\alpha; \alpha \in \xi\})$ does not depend on $\mathbb{Q}(p,q,r)$ but only on $\{g_\alpha; \alpha \in \xi\}$; however the partial order $\mathbb{R}(\{g_\alpha; \alpha \in \xi\})$ will only be used in the context that $\{g_\alpha;\alpha\in\xi\}\subseteq\mathbb{Q}(p,q,r)$ for some p, q and r. The following lemma illustrates how these partial orders will be used to construct conditions in the partial order $\mathbb{Q}(p,q,r)$.

LEMMA 4.1: *Suppose that:*

1. *p and q are $P_{\mathfrak{c}}$-points in $\beta\mathbb{N}\setminus\mathbb{N}$,*
2. *r is a filter such that $p\bowtie_r q$,*
3. *$\{g_\xi;\xi\in\eta\}$ is an increasing sequence from $\mathbb{Q}(p,q,r)$,*
4. *$\mathbb{R}(\{g_\xi;\xi\in\eta\})\in\mathfrak{A}$ and $\mathfrak{A}$ is a countable elementary submodel of $H(\omega_2)$.*

Then there is $g\in\mathbb{Q}(p,q,r)$ which is $\mathfrak{A}$-generic for $\mathbb{R}(\{g_\xi;\xi\in\eta\})$. Moreover, for any extension $\{g_\xi;\xi\in\mu\}$ of $\{g_\xi;\xi\in\eta\}$ such that $g_\eta=g$, every $D\in\mathfrak{A}$ is predense in $\mathbb{R}(\{g_\xi;\xi\in\mu\})$ provided that it is dense in $\mathbb{R}(\{g_\xi;\xi\in\eta\})$.

Proof: Let $\{D_n;n\in\omega\}$ enumerate the dense subsets of $\mathbb{R}(\{g_\xi;\xi\in\eta\})$ in $\mathfrak{A}$. Players I and II will now play the game $G(p,q,r)$ with Player I using the strategy about to be described. At stage n of the game suppose that Player II has chosen $\{(a_i,b_i,c_i);i\in n\}$ while Player I has chosen $\{(A_i,B_i,C_i);i\in n\}$ as well as conditions $\{h_i;i\in n\}$ from $\mathbb{R}(\{g_\xi;\xi\in\eta\})$. Then Player I's strategy is to choose A_n, B_n and C_n according to the following plan:

1. choose an arbitrary enumeration $\{(g^i_n,s^i);i\in M_n\}$ of all pairs (g,s) such that $g:c_{n-1}\to c_{n-1}$ is a bijection and $s\in n$;
2. then choose a sequence $\{f^i_n;i\in M_n\}\subseteq\mathbb{Q}(p,q,r)$ such that for each i the following conditions hold:
   (a) $f^i_n\in D_{s^i}$ and $g^i_n\subseteq f^i_n$,
   (b) $f^i_n\restriction(\omega\setminus c_{n-1})\subseteq f^{i+1}_n$,
   (c) $\bigcup\{h_i;i\in n\}\restriction(\omega\setminus c_{n-1})\subseteq f^0_n$;
3. then let $C_n=(\bigcap\{\Sigma(f^i_n);i\in M_n\}\setminus c_{n-1})\cap(\bigcap\{C_i;i\in n\})$;
4. define $h_n=(\bigcup\{f^i_n\restriction(\omega\setminus c_{n-1});i\in M_n\})\cup(\bigcup\{h_i;i\in n\})$;
5. let $A_n=\omega\setminus\mathrm{dom}(h_n)$ and $B_n=\omega\setminus\mathrm{ran}(h_n)$.

It is easy to check that the density of D_{s^i} ensures that it is always possible for Player I to follow this strategy. The main thing to notice is that, because $c_{n-1}\in\bigcap\{C_i;i\in n\}$ it follows that $h_i(j)\in c_{n-1}$ and $h_i^{-1}(j)\in c_{n-1}$ for each

$j \in c_{n-1}$ and $i \in n$ — hence there is no conflict, in defining f_n^0 to be a one-to-one function, with the restriction that $g_n^0 \subseteq f_n^0$ and $\cup\{h_i; i \in n\} \restriction (\omega \setminus c_n) \subseteq f_n^0$. In other words, $g_n^i \cup (\cup\{h_i; i \in n\}) \restriction (\omega \setminus c_{n-1})$ is a one-to-one function for each $i \in M_n$.

By the hypothesis on p, q and r this is not a winning strategy and so there must be some play of the game $\{(A_k, B_k, C_k), (a_k, b_k, c_k); k \in \omega\}$ in which Player I has played according to this strategy but, nevertheless, Player II has won. In particular, $\cup\{a_j; j \in \omega\} = A \in p$, $\cup\{b_j; j \in \omega\} = B \in q$ and $\{c_j; j \in \omega\} = C \in r$. Notice that $a_j \cap \text{dom}(h_i) = \emptyset$ for all i and j because if $i \leq j$ then $a_j \subseteq A_j = \omega \setminus \text{dom}(h_j) \subseteq \omega \setminus \text{dom}(h_i)$ while if $i > j$ then $\text{dom}(h_i) \cap c_j = \emptyset$ and $c_j > \max(a_j)$ because of the rules of the game $G(p, q, r)$; hence $\text{dom}(h_i) \cap A = \emptyset$. Similarly, $\text{ran}(h_i) \cap B = \emptyset$ for each $i \in \omega$. It follows that if g is defined to be $\cup\{h_k; k \in \omega\}$ then $g \in \mathbb{Q}(p, q, r)$ provided that it can be shown that $\Sigma(g) \supseteq C$. To see this let, $c_n \in C$. Observe that $c_n \in \Sigma(\cup_{i \in n+1} h_i)$ by definition. Also note that, since Player I's strategy involved enumerating only bijections in $\{(g_{n+1}^i, s^i); i \in M_{n+1}\}$, it follows that $c_n \in \Sigma(f_{n+1}^i)$ for each $i \in M_{n+1}$ and hence

$$\text{dom}(h_{n+1} \setminus \cup_{i \in n+1} h_i) \cup \text{ran}(h_{n+1} \setminus \cup_{i \in n+1} h_i) \subseteq \omega \setminus c_n.$$

Furthermore, if $m > n + 1$ then $\text{dom}(h_m) \cup \text{ran}(h_m) \subseteq \omega \setminus c_n$. From these facts it follows that $\Sigma(g) \supseteq C$.

Moreover, g is $\mathfrak{A}$-generic for $\mathbb{R}(\{g_\xi; \xi \in \eta\})$ in the strong sense of the lemma. To see this suppose that $D \in \mathfrak{A}$ is a dense set in $\mathbb{R}(\{g_\xi; \xi \in \eta\})$ and $\mathbb{R}(\{g_\xi; \xi \in \mu\})$ is a sequence extending $\mathbb{R}(\{g_\xi; \xi \in \eta\})$ such that $g = g_\eta$. Let K be such that $D = D_K$. Suppose also that $g \subseteq h$, $h \in \mathbb{R}(\{g_\xi; \xi \in \mu\})$ and that h is incompatible with every element of D. Since $h \in \mathbb{R}(\{g_\xi; \xi \in \mu\})$ it follows that $\Sigma(h) \in r$ and hence there is some $k \in C \cap \Sigma(h)$ such that $|C \cap k| \geq K$. Hence, for some j, $k = c_j$ was chosen by Player II at stage j of the game and $j \geq K$.

Now consider the strategy used by Player I to choose A_{j+1}, B_{j+1} and C_{j+1}. In the enumeration $\{(g_{j+1}^i, s^i); i \in M_{j+1}\}$ of the subset of ${}^{c_j}c_j \times (j+1)$, whose first coordinates are bijections, there is some $m \in M_{j+1}$ such that $g_{j+1}^m \supseteq h \restriction c_j$ and $s^m = K$. It follows that $f_{j+1}^m \in D$ by Player I's strategy. Also $f_{j+1}^m \restriction (\omega \setminus c_j) \subseteq h_{j+1} \subseteq g \subseteq h$ and $f_{j+1}^m \restriction c_j \supseteq h \restriction c_j$. Moreover, because $c_j \in \Sigma(h)$ it follows that $h \restriction (\omega \setminus c_j) \cup g$ is one-to-one for any one-to-one $g : c_j \to c_j$. It follows that h is compatible with some member of D — namely f_{j+1}^m. ∎

Lemma 4.2: *$\mathbb{P}_{\omega_3}$ is $\aleph_2$-distributive.*

Proof: Let $\{E_\mu; \mu \in \omega_1\}$ be dense sets in $\mathbb{P}_{\omega_3}$ and and let $f \in \mathbb{P}_{\omega_3}$. Let $\mathbb{D}$ be a countably closed forcing notion which adds a $\Diamond$-sequence $\{A_\alpha; \alpha \in \omega_1\}$ — to be precise, $\{A_\alpha; \alpha \in \omega_1\}$ is a collection such that $A_\alpha \in [\alpha]^{\aleph_0}$ for each $\alpha \in \omega_1$ and, moreover, there is a surjection $\Theta : \omega_1 \mapsto \omega_1$ such that for all $X \subseteq H_{\omega_1}$ and $\beta \in \omega_1$ the set $\{\mu \in \omega_1; X \cap \mu \in A_\alpha \text{ and } \Theta(\mu) = \beta\}$ is stationary. In the model obtained by forcing with $\mathbb{D}$ let $\mathfrak{M} \prec H_{\omega_4}$ be such that $|\mathfrak{M}| = \aleph_1$, $[\mathfrak{M}]^{\aleph_0} \subseteq \mathfrak{M}$ and both f and $\mathbb{P}_{\omega_3}$ belong to $\mathfrak{M}$ as do the indexed families $\{A_\alpha; \alpha \in \omega_1\}$ and $\{E_\alpha; \alpha \in \omega_1\}$. Notice that ω_1 is both an element as well as a subset of $\mathfrak{M}$. Hence each individual A_α and E_α belongs to $\mathfrak{M}$.

By extending the sets A_α for $\alpha \in \omega_1$ it may, without loss of generality, be assumed that

- each A_α is a model of a large fragment of set theory,
- $A_\alpha \subseteq A_\beta$ if $\alpha \in \beta$,
- $\bigcup\{A_\alpha; \alpha \in \omega_1\} \supseteq \mathfrak{M} \cap H_{\omega_1}$.

Let $\theta : \omega_1 \mapsto [\mathfrak{M} \cap \omega_3]^{<\aleph_0}$ be a surjection such that $\Theta(\alpha) = \Theta(\beta)$ implies that $\theta(\alpha) = \theta(\beta)$.

To finish the proof assume that a sequence $\{f_\alpha; \alpha \in \omega_1\}$ has been constructed such that the following properties are satisfied:

1. $f_0 = f$,
2. $f_{\alpha+1} \in E_\alpha$,
3. $f_\alpha \in \mathbb{P}_{\omega_3} \cap \mathfrak{M}$,
4. if $\alpha \in \beta$ then $f_\alpha \geq f_\beta$,
5. for all $\mu \in \bigcup_{\alpha\in\omega_1}\text{support}(f_\alpha)$ there is $\alpha(\mu)$ such that for each $\alpha > \alpha(\mu)$ there is g^α_μ such that $f_\alpha \restriction \mu \Vdash_{\mathbb{P}_\mu}$ "$f_\alpha(\mu) = \check{g}^\alpha_\mu$" — if $\alpha \leq \alpha(\mu)$ then define $g^\alpha_\mu = \emptyset$,
6. if $\gamma \in \theta(\alpha)$ then

$$f_\alpha \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“} f_\alpha(\gamma) \text{ is } A_\alpha[\{f_\alpha(\delta); \delta \in \theta(\alpha) \cap \gamma\}]\text{-generic for } \mathbb{R}(\{g^\zeta_\gamma; \zeta \in \alpha\})\text{”}.$$

Such a sequence is easily constructed using Lemma 4.1.

Given such a sequence $\{f_\alpha; \alpha \in \omega_1\}$ it follows from condition 6 and the hypothesis on Θ and θ that for each $\Gamma \in [\mathfrak{M} \cap \omega_3]^{<\aleph_0}$

$$\prod_{\gamma \in \Gamma} \mathbb{R}(\{g^\alpha_\gamma; \alpha \in \omega_1\})$$

has the countable chain condition. (Note that the iteration *can* be written as a product because the partial orders $\mathbb{R}(\{g^\alpha_\gamma; \alpha \in \omega_1\})$ are in the ground model since the sequence $\{g^\alpha_\gamma; \alpha \in \omega_1\}$ has been decided and no new reals are added by forcing with $\mathbb{P}_\mu$. Also, the ultrafilters p_γ, q_γ and r_γ play no role in the definition of the partial order $\mathbb{R}(\{g^\alpha_\gamma; \alpha \in \omega_1\})$). The standard argument for this is to suppose that

$$C \subseteq \prod_{\gamma \in \Gamma} \mathbb{R}(\{g^\alpha_\gamma; \alpha \in \omega_1\})$$

is a maximal antichain for some $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \in [\mathfrak{M} \cap \omega_3]^{<\aleph_0}$. The hypothesis on $\{A_\xi; \xi \in \omega_1\}$ ensures that there is some $\xi \in \omega_1$ such that

- $\xi > \alpha(\gamma)$ for $\gamma \in \Gamma$,
- $\theta(\xi) = \Gamma \in A_\xi$,
- $\{g^\alpha_\gamma; \alpha \in \xi\} \in A_\xi$,
- $C \cap \prod_{\gamma \in \Gamma} \mathbb{R}(\{g^\alpha_\gamma; \alpha \in \xi\}) \in A_\xi$ is a maximal antichain in

$$\prod_{\gamma \in \Gamma} \mathbb{R}(\{g^\alpha_\gamma; \alpha \in \xi\}).$$

Since $f_\xi(\gamma)$ is $A_\xi[\{f_\xi(\delta); \delta \in \gamma \cap \theta(\xi)\}]$-generic for the partial order $\mathbb{R}(\{g^\zeta_\gamma; \zeta \in \xi\})$ it follows that $(f_\xi(\gamma_1), f_\xi(\gamma_2), \ldots, f_\xi(\gamma_k))$ is A_ξ-generic for the partial order

$$\prod_{i=1}^{m} \mathbb{R}(\{g^\alpha_{\gamma_i}; \alpha \in \omega_1\}).$$

This follows from the general fact [3] that if p is $\mathfrak{A}$-generic for $\mathbb{P}$ and

$$p \Vdash_{\mathbb{P}} \text{“}q \text{ is } \mathfrak{A}[G]\text{-generic for } \mathbb{Q}\text{”}$$

then $p * q$ is $\mathfrak{A}$-generic for $\mathbb{P} * \mathbb{Q}$.

Hence $C \cap \prod_{\gamma \in \Gamma} \mathbb{R}(\{g_\gamma^\alpha; \alpha \in \xi\})$ is a maximal antichain in

$$\prod_{i=1}^{m} \mathbb{R}(\{g_{\gamma_i}^\alpha; \alpha \in \omega_1\})$$

because of the genericity of $(f_\xi(\gamma_1), f_\xi(\gamma_2), \ldots, f_\xi(\gamma_k))$ over the model A_ξ — this relies on the fact that $\theta(\xi) = \Gamma$ — and the fact $C \cap \prod_{\gamma \in \Gamma} \mathbb{R}(\{g_\gamma^\alpha; \alpha \in \xi\}) \in A_\xi$. Hence C is countable and it follows that

$$\prod_{\gamma \in \mathfrak{M} \cap \omega_3} \mathbb{R}(\{g_\gamma^\alpha; \alpha \in \omega_1\})$$

has the countable chain condition since each finite subproduct does.

Let G be

$$\prod_{\gamma \in \mathfrak{M} \cap \omega_3} \mathbb{R}(\{g_\gamma^\alpha; \alpha \in \omega_1\})$$

generic and define $f'_{\omega_1}(\gamma) = \cup\{g(\gamma); g \in G\}$. It is clear that $f'_{\omega_1}(\gamma) \supseteq^* f_\alpha(\gamma)$ for each $\alpha \in \omega_1$ and $\gamma \in \mathfrak{M} \cap \omega_3$ and that $f'_{\omega_1}(\gamma)$ is a one-to-one function. What is not yet true is that $\mathrm{dom}(f'_{\omega_1}(\gamma)) \in p_\gamma$ and $\mathrm{ran}(f'_{\omega_1}(\gamma)) \in p_\gamma$. However to obtain the structure $(\{f_\xi; \xi \in \omega_1\}, f')$ requires meeting only $\aleph_1$ dense sets in

$$\mathbb{D} * \prod_{\gamma \in \omega_1} \mathbb{R}(\{g_\gamma^\alpha; \alpha \in \omega_1\})$$

and since this is a proper partial order it follows from PFA that the structure can be constructed. It then follows from Lemma 3.1 and the fact that

$$\emptyset \Vdash_{\mathbb{P}_\eta} \text{“}p_\eta \text{ and } q_\eta \text{ are } P_\mathfrak{c}\text{-points and } r_\eta \text{ is a } P_\mathfrak{c}\text{-filter”}$$

that there is a $f \in \mathbb{P}_{\omega_3}$ such that $\mathrm{dom}(f) = \mathfrak{M} \cap \omega_3$ and

$$f \restriction \eta \Vdash_{\mathbb{P}_\eta} \text{“}f(\eta) \subseteq f'(\eta) \text{ and } f(\eta) \supseteq^* f_\mu(\eta)\text{”}$$

for each $\eta \in \mathfrak{M} \cap \omega_3$ and $\mu \in \omega_1$. Note that, in general, $f(\eta)$ is a $\mathbb{P}_\eta$-name for a condition and not of the form $\check{g}$ because the filters p_η, q_η and r_η may not have been determined. ∎

The results of this section prove the following theorem.

THEOREM 4.1: *If ZF is consistent then so is ZFC and MA_{ω_1} and every two $P_\mathfrak{c}$-points are topologically equivalent.*

Proof: Let V be a model of PFA and let G be $\mathbb{P}_{\omega_3}$-generic over V. In the resulting model MA_{ω_1} still holds and also if p and q are any two $P_{\mathfrak{c}}$-points such that $p \bowtie q$ then p and q have the same topological type. The result then follows because Lemma 2.2 implies that if p and q are any two $P_{\mathfrak{c}}$-points then $p \bowtie q$. ∎

5. Open questions

It has been shown that it is consistent with MA_{ω_1} that there are topological equivalence classes both of size $2^{\aleph_0}$ and $2^{2^{\aleph_0}}$ but are these the only possibilities?

QUESTION 5.1: *Is it consistent with MA that there is a topological equivalence class of size κ and $2^{\aleph_0} < \kappa < 2^{2^{\aleph_0}}$?*

QUESTION 5.2: *Is it consistent with ZFC that there is a topological equivalence class of size κ and $2^{\aleph_0} < \kappa < 2^{2^{\aleph_0}}$?*

QUESTION 5.3: *Is it consistent with MA that there are two topological equivalence classes of different cardinalities?*

The question of the number of different topological types in $\beta\mathbb{N} \setminus \mathbb{N}$ still does not seem to be completely understood.

QUESTION 5.4: *Is it consistent with MA that there are less than $2^{2^{\aleph_0}}$ topological equivalence classes in $\beta\mathbb{N} \setminus \mathbb{N}$?*

Theorem 4.1 refers only to $P_{\mathfrak{c}}$-points but in order to answer Question 5.4 it will be necessary to obtain similar results for other points as well. The following question indicates only one of several possibilities.

QUESTION 5.5: *Is it consistent with MA_{ω_1} that any two P_{ω_1}-points are topologically equivalent?*

QUESTION 5.6: *If the answer to Question 5.5 is negative then what is the best that can be done?*

It is worth noting that the homeomorphisms constructed in Theorem 4.1 are all of a special type — in the sense that they can be considered to be limits of trivial homeomorphisms. Is it consistent with MA that there is a homeomorphism which is not of this type? Perhaps MA implies a structure theorem for homeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N}$.

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